Lecture 16 :The Mean Value Theorem

We know that constant functions have derivative zero. Is it possible for a more complicated function to have derivative zero?
In this section we will answer this question and a related question: How are two functions with the same derivative related?
Below we look at two important theorems which give us more information on the behavior of a continuous function on a closed interval $[a, b]$, when we add the extra assumption that the function is differentiable on the open interval (a, b).

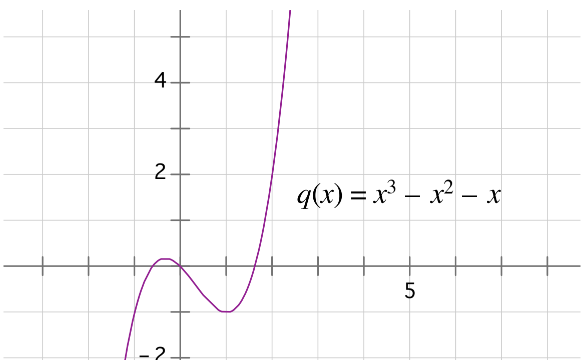
Rolle's Theorem

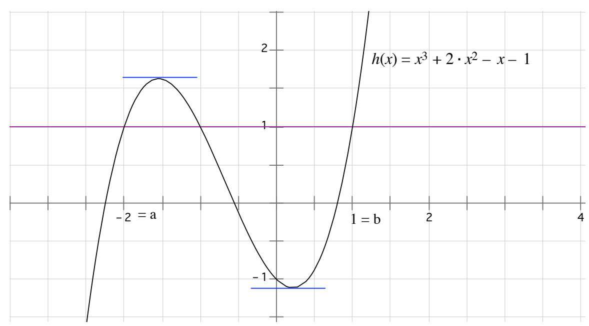
Rolle's Theorem Suppose that

- $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and
- differentiable at every point of its interior (a, b) and
- $f(a) = f(b)$,

then there is at least one point c in (a, b) at which $f'(c) = 0$.

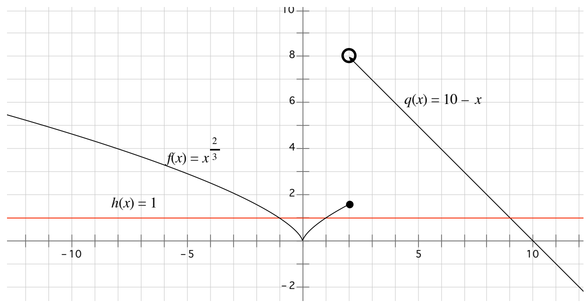
The graphs of some functions satisfying the hypotheses of the theorem are shown below:





Let $q(x) = x^3 - x^2 - x$. The graph $y = q(x)$ is shown on the left above. We see that $q(-1) = -1 = q(1)$. Since $q(x)$ is a polynomial it is continuous on the interval $[-1, 1]$ and differentiable on the the open interval $(-1, 1)$.
Therefore Rolle's theorem applies and we know that there is some number c with $-1 < c < 1$ for which $q'(c) = 0$. That is the tangent to the graph is horizontal at c. (See the corresponding turning point on the graph above).
We can make similar conclusions for the function $h(x) = x^3 + 2x^2 - x - 1$, the graph of which is shown on the right above.
Example Let $h(x) = x^3 + 2x^2 - x - 1$. Find a number c such that $-2 < c < 1$ so that the tangent to the graph of $y = h(x)$ is horizontal at $x = c$.
Since $h(-2) = 1 = h(1)$ and $h(x)$ is continuous at every point of the interval $[-2, 1]$ and differentiable at every point of its interior $(-2, 1)$, Rolles Theorem guarantees that there is at least one point c in the interval $(-2, 1)$ with $h'(c) = 0$.
Here, we have $h'(x) = 3x^2 + 4x - 1$. To find a value of c with $-2 < c < 1$ such that $h'(c) = 0$, we must solve the quadratic $3c^2 + 4c - 1 = 0$. We get $c = \dfrac{-4 \pm \sqrt{16 + 12}}{6} = \dfrac{-2 \pm \sqrt{7}}{3} \approx -1.55,\ .215.$
Note: Finding or estimating the value of such a c is not always easy.

Note the result does not always work if one of the conditions above is violated. Note that in the graph of the piecewise defined function $h(x)$ below, we have $h(-1) = 1 = h(1) = h(9)$. However there are no values of c with $h'(c) = 0$ (horizontal tangent) on the graph. Why does Rolle's theorem not apply here?



Example: Movement of a particle If $s = f(t)$ is a smooth function describing the position of an object in a straight line. If the object is in the same position at times $t = a$ and $t = b$ then $f(a) = f(b)$ and by Rolle's theorem there must be a time c in between when $v(c) = f'(c) = 0$, that is the object comes to rest.

Using Rolles Theorem With The intermediate Value Theorem

Example Consider the equation $x^3 + 3x + 1 = 0$. We can use the Intermediate Value Theorem to show that has at least one real solution:
If we let $f(x) = x^3+3x+1$, we see that $f(-1) = -3 < 0$ and $f(1) = 5 > 0$. Since $f(x)$ is a polynomial, it is continuous everywhere and the **Intermediate Value Theorem guarantees** that there is a number c with $-1 < c < 1$ for which $f(c) = 0$ (in other words c is a **root of the equation** $x^3 + 3x + 1 = 0$).
We can **use Rolle's Theorem to show that there is only one real root of this equation.**

Proof by Contradiction Assume Statement X is true. Show that this leads to a contradiction. Conclusion: Statement X cannot be true.

The Mean Value Theorem

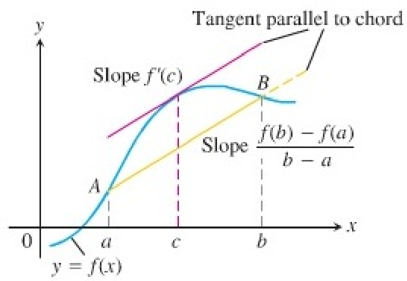
This is a slanted version of Rolle's theorem:

Mean Value Theorem Suppose

- $y = f(x)$ is continuous on a closed interval $[a, b]$ and
- differentiable on the interval's interior (a, b).

Then there is at least one point c in (a, b) where

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad \text{or} \qquad f(b) - f(a) = f'(c)(b - a).$$



Geometrically the mean value theorem says that somewhere between A and B, the graph has a tangent parallel to the chord(secant) AB.

Physical Interpretation Recall $\frac{f(b)-f(a)}{b-a}$ is the average rate of change of the function f on the interval $[a, b]$ and $f'(c)$ is the instantaneous rate of change at the point c. *The Mean Value Theorem says that at some point in the interval $[a, b]$ the instantaneous rate of change is equal to the average rate of change over the interval (as long as the function is continuous on $[a, b]$ and differentiable on (a, b).)*

Sometimes we can find a value of c that satisfies the conditions of the mean Value Theorem.

Example Let $f(x) = x^3 + 2x^2 - x - 1$, find all numbers c that satisfy the conditions of the Mean Value Theorem in the interval $[-1, 2]$.

f is continuous on the closed interval $[-1, 2]$ and differentiable on the open interval $(-1, 2)$. Therefore the Mean Value theorem applies to f on $[-1, 2]$.

The value of $\dfrac{f(b) - f(a)}{b - a}$ here is :

Fill in the blanks: The Mean Value Theorem says that there exists a (at least one) number c in the interval _______ such that $f'(c) =$ ______.

To find such a c we must solve the equation ______

Example A car passes a camera at a point A on the toll road with speed 50 mph. Sixty minutes later the same car passes a camera at a point B, located 100 miles down the road from camera A, traveling at 50 mph. Can we prove that the car was breaking the speed limit (75 m.p.h.) at some point along the road?

We can also use this theorem to make inferences about the growth of a function from knowledge about its derivative:

Example If $f(0) = 1$, $f'(x)$ exists for all values of x and $f'(x) \leq 4$ for all x, how large can $f(2)$ possibly be?

Example If $f(0) = 5$, $f'(x)$ exists for all x and $-1 \leq f'(x) \leq 3$ for all x, show that

$$-5 \leq f(10) \leq 35$$

Mathematical Consequences

With the aid of the Mean Value Theorem we can now answer the questions we posed at the beginning of the section.

Consequence 1 If $f'(x) = 0$ at each point in an open interval (a, b), we can conclude that $f(x) = C$ for some constant C for all x in the interval (a, b).

Proof Let x_1 and x_2 be points in the interval $[a, b]$ with $x_1 < x_2$. Now since $f'(x) = 0$ in the interval (a, b), we know that f is continuous on $[x_1, x_2]$ and differentiable on the interval (x_1, x_2). Therefore the Mean Value theorem applies and

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

for some c with $x_1 < c < x_2$. All such $f'(c)$ equal 0, therefore

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$$

This gives

$$f(x_1) - f(x_2) = 0(x_1 - x_2) = 0 \quad \text{and} \quad f(x_1) = f(x_2).$$

This works for all x_1 and x_2 in the interval (a, b), hence $f(x)$ is constant on the interval.

Consequence 2 If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is $f - g$ is a constant function.

Example Find a function with $f'(x) = 3x^2$ that runs through the point $(0, 1)$.

Solution We must have $f(0) = 1$, since the function passes through the point $(0, 1)$.
Now it is not difficult to find a function $F(x)$ with $F'(x) = 3x^2$ by guessing. In fact $F(x) = x^3$ will work.
By Consequence 2 above, the function $f(x)$ that we are searching for must have a formula $f(x) = F(x) + C = x^4 + C$ for some constant C, since $f'(x) = 3x^2 = F'(x)$.
Now using the fact that $f(0) = 1$, we solve for C: $f(0) = 0^4 + C$ implies that $C = 1$ and

$$f(x) = x^4 + 1.$$

Rolle's Theorem Suppose that

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- differentiable at every point of its interior (a, b) and
- $f(a) = f(b)$,

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Proof of Rolle's Theorem: Because f is continuous on the closed interval $[a, b]$, f attains maximum and a minimum value on $[a, b]$. This maximum can occur at

1. at interior points where f' is zero
2. at interior points where f' does not exist
3. at the endpoints of the interval, a or b.

We are assuming that f has a derivative at every interior point. This rules out the possibility of 2, leaving us with interior points where f' is zero and two endpoints a and b.

If either the maximum or the minimum occurs at a point c between a and b where $f'(c) = 0$, then we have found a point c for Rolle's theorem.

If both the absolute maximum and minimum occur at $f(a)$ and $f(b)$, then, since $f(a) = f(b) = k$, when $a < x < b$ we cannot have $f(x) < f(a)$, nor can we have $f(x) > f(a)$. Therefor the function must be constant on the interval $[a, b]$, that is $f(x) = k$ for all $a \leq x \leq b$. Hence at every point in the interior of the interval, we have $f'(x) = 0$ and Rolle's theorem holds.

The Mean Value Theorem

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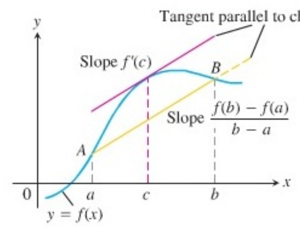


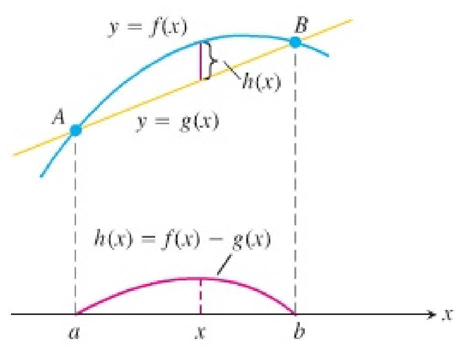
Geometrically the mean value theorem says that somewhere between A and B, the graph has a tangent parallel to the chord(secant) AB.

Proof The chord AB coincides with the graph of the function

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

Now consider the function

$$h(x) = f(x) - g(x).$$



We have that $h(x)$ is contionuous on $[a, b]$ and differentiable on (a, b), because both $f(x)$ and $g(x)$ are. Also $h(a) = 0$ and $h(b) = 0$, hence we can apply Rolle's theorem to show that there is a point c in the interval (a, b) at which $h'(c) = 0$. This gives $f'(c) - g'(c) = 0$ or $f'(c) = g'(c)$. Thus we get:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

and we are done with the proof.